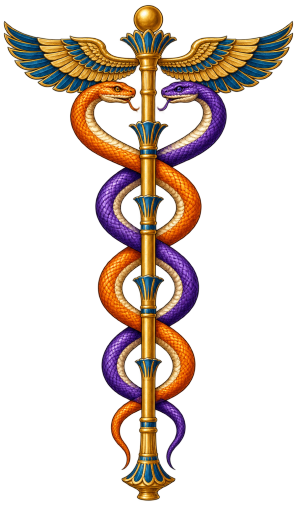
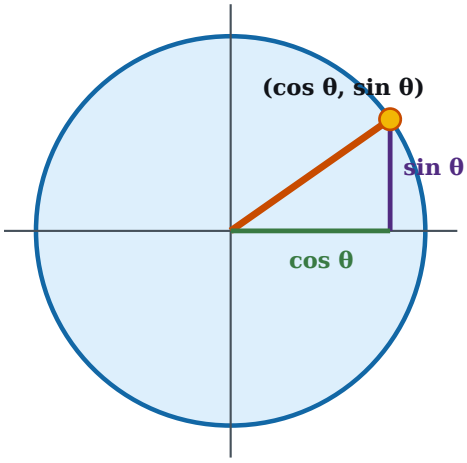
FINDING SIN, COS & TAN OF THETA

& CORRESPONDING INVERSE FUNCTIONS

By Hand With No Calculators



THE UNIT CIRCLE



sin θ = y/r • cos θ = x/r • tan θ = y/x

CORE PROMISE
The book develops exact values, estimates, inverse angles, identities, and checks using diagrams, fractions, radicals, written multiplication, and long division rather than modern electronic calculators.

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About This Textbook

This textbook teaches how to determine sin θ, cos θ, tan θ, arcsin, arccos, and arctan without depending on a modern calculator. It combines right-triangle reasoning, the unit circle, exact special angles, identities, reference angles, written series, rational approximations, interpolation, and independent checks.

EXACT BEFORE DECIMAL
Whenever possible, keep √2/2, √3/2, 2−√3, and similar expressions exact. Convert to a decimal only when measurement or comparison requires one, and report no more digits than the hand method supports.

Learning Objectives

- Convert angles between degrees, radians, revolutions, and quadrants.
- Use SOH-CAH-TOA and the unit circle to define sin, cos, and tan.
- Derive the exact values for 0°, 30°, 45°, 60°, and 90°.
- Find values in all quadrants using reference angles and sign rules.
- Apply Pythagorean, reciprocal, quotient, cofunction, sum, difference, double-angle, and half-angle identities.
- Build and use hand tables through recurrence, interpolation, Bhaskara's approximation, and power series.
- Distinguish inverse trigonometric functions from reciprocal functions.
- Find exact and approximate inverse values while respecting principal ranges.
- Solve right triangles and basic trigonometric equations by hand.
- Explain the caduceus and helix analogy precisely, including its limitations.

How to Use the Book

Feature	Purpose	Recommended Practice
Exact-value tables	Preserve complete angle-ratio relationships.	Recreate the table from triangles rather than memorizing blindly.
Method boxes	Give repeatable hand procedures.	Copy each method with one worked example.
Error bounds	Show which decimal digits are justified.	Square, substitute, or compare neighboring table values.
Historical article	Connect algorithms to pre-electronic computation.	Separate documented history from modern visual analogy.
Problem bank and solutions	Build fluency and self-checking.	Show exact work before any approximation.

Notation and Angle Mode

Unless a degree symbol appears, angles in formulas and series are measured in radians. The notation $\sin^{-1}x$ means arcsin x, not 1/sin x. Reciprocal sine is csc x. Likewise $\cos^{-1}x=\arccos x$, while sec x=1/cos x.

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TEXTBOOK ORGANIZATION
Part I builds angle and unit-circle foundations. Part II develops exact and approximate hand methods. Part III treats inverse functions and equations. Part IV presents the caduceus, helix, and historical computation articles. Part V supplies applications, investigations, assessment, solutions, references, and an index.

Quick Diagnostic

No.	Question	No.	Question
1	Convert 150° to radians.	6	Find sin 30° exactly.
2	Convert 7π/4 to degrees.	7	Find tan 45° exactly.
3	State SOH-CAH-TOA.	8	Determine the sign of cos 120°.
4	What point corresponds to θ on the unit circle?	9	Evaluate arcsin(1/2).
5	State $\sin^2\theta+\cos^2\theta$.	10	Explain why $\sin^{-1}\theta$ is not csc θ.

Answers appear in Chapter 22. Use errors to select the chapters needing the most attention.

PART I • FOUNDATIONS

1. Angles, Theta, Degrees, and Radians

The language required before any trigonometric computation

1.1 Theta Is an Angle Variable

The Greek letter θ is commonly used for an unknown or changing angle. An angle measures rotation from an initial ray to a terminal ray. Counterclockwise rotation is conventionally positive; clockwise rotation is negative.

1.2 Degrees, Revolutions, and Radians

360° = 1 revolution = 2π radians

Convert	Multiply By	Example
Degrees to radians	π/180	150°·π/180=5π/6
Radians to degrees	180/π	7π/4·180/π=315°
Revolutions to degrees	360	3/4 revolution=270°
Revolutions to radians	2π	3/4 revolution=3π/2

1.3 Why Radians Matter

A radian measures arc length divided by radius: θ=s/r. It links circular geometry directly to algebra and makes the derivative and power-series formulas for sine and cosine take their simplest forms.

1.4 Coterminal and Reference Angles

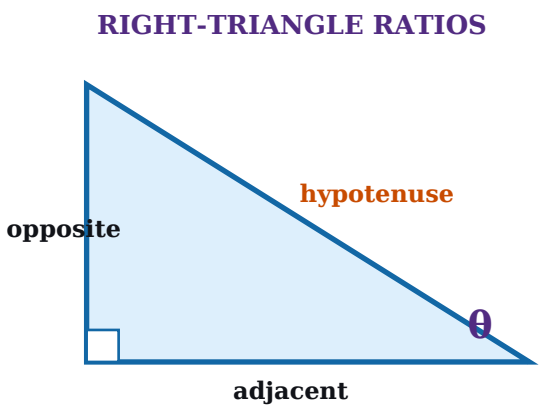
Angles differing by full revolutions share a terminal side: θ+360°k or θ+2πk. The reference angle is the positive acute angle between the terminal side and the x-axis. It determines the magnitude of trigonometric values; the quadrant determines the sign.

Quadrant	Degree Interval	Reference Angle α
I	0° to 90°	α=θ
II	90° to 180°	α=180°−θ
III	180° to 270°	α=θ−180°
IV	270° to 360°	α=360°−θ

PART I • FOUNDATIONS

2. Right-Triangle Definitions

SOH-CAH-TOA, Pythagorean triples, and ratio reasoning



For an acute angle θ in a right triangle:

sin θ = opposite / hypotenuse
cos θ = adjacent / hypotenuse
tan θ = opposite / adjacent

These ratios depend on the angle, not on the triangle's overall size, because similar triangles preserve ratios.

2.1 SOH-CAH-TOA

Mnemonic	Ratio	Reciprocal
SOH	sin θ=opp/hyp	csc θ=hyp/opp
CAH	cos θ=adj/hyp	sec θ=hyp/adj
TOA	tan θ=opp/adj	cot θ=adj/opp

2.2 Example: A 3-4-5 Triangle

Let θ be opposite the side of length 3 and adjacent to the side of length 4. The hypotenuse is 5. Therefore $\sin\theta=3/5$, $\cos\theta=4/5$, and $\tan\theta=3/4$. The ratios verify $\tan\theta=\sin\theta/\cos\theta$.

2.3 Recovering a Missing Side

If two sides are known, use $a^2+b^2=c^2$ before forming ratios. Common triples reduce hand work: 3-4-5, 5-12-13, 8-15-17, 7-24-25, and their multiples.

2.4 Limits of the Triangle Definition

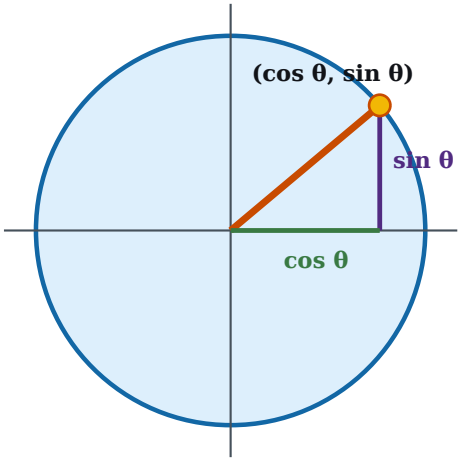
A right triangle directly represents acute angles. The unit-circle definition extends sin, cos, and tan to obtuse, negative, and multi-revolution angles while preserving the same ratios in Quadrant I.

PART I • FOUNDATIONS

3. The Unit Circle and Coordinate Definitions

One circle organizes signs, exact values, periodicity, and graphs

THE UNIT CIRCLE



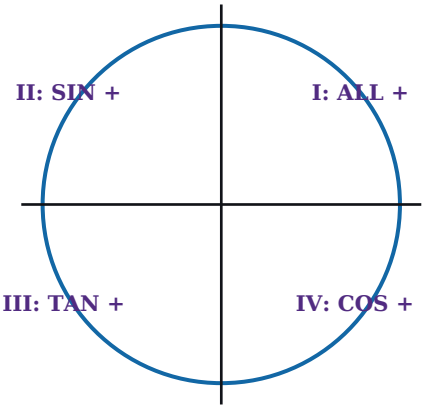
On a circle of radius r, a terminal point (x,y) gives:

$\sin\theta=y/r$
$\cos\theta=x/r$
$\tan\theta=y/x$ when $x\neq0$.

On the unit circle r=1, the point is exactly $(\cos\theta,\sin\theta)$.

3.1 Signs by Quadrant

QUADRANT SIGN MAP



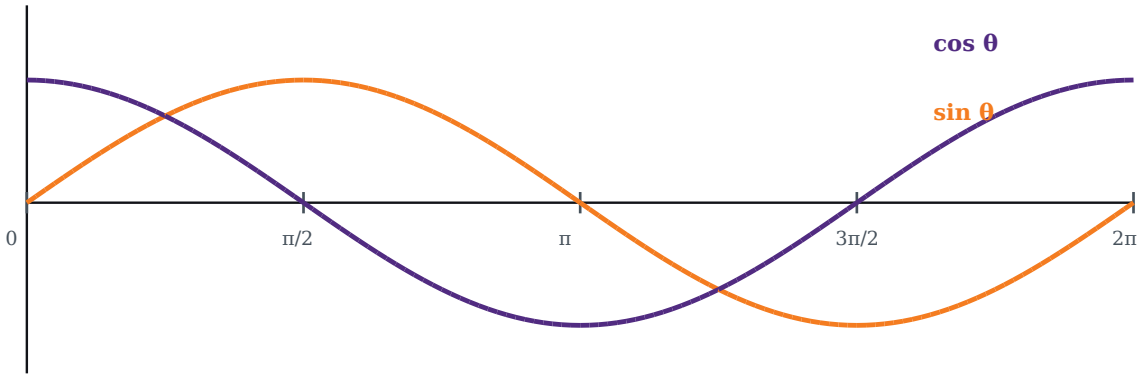
Quadrant	sin	cos	tan
I	+	+	+
II	+	−	−
III	−	−	+
IV	−	+	−

3.2 Periods

$$\sin(\theta+2\pi)=\sin\theta,\ \cos(\theta+2\pi)=\cos\theta,\ \tan(\theta+\pi)=\tan\theta$$

3.3 Graphs From Circular Motion

SINE AND COSINE: SAME SHAPE, QUARTER-CYCLE SHIFT

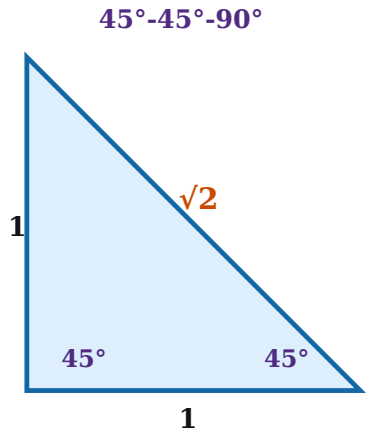


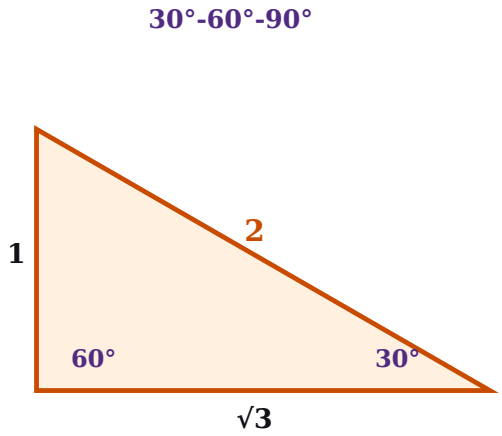
As a point rotates uniformly, its vertical coordinate traces sine and its horizontal coordinate traces cosine. Cosine leads sine by π/2: $\cos\theta=\sin(\theta+\pi/2)$. Tangent equals their quotient and is undefined wherever $\cos\theta=0$.

PART I • FOUNDATIONS

4. Exact Special Angles

Derive 30°, 45°, and 60° from geometry





4.1 The 45°-45°-90° Triangle

Give both legs length 1. The hypotenuse is $\sqrt{(1^2+1^2)}=\sqrt{2}$. Therefore $\sin45°=\cos45°=1/\sqrt{2}=\sqrt{2}/2$ and $\tan45°=1$.

4.2 The 30°-60°-90° Triangle

Bisect an equilateral triangle of side 2. The half-base is 1, the hypotenuse is 2, and the height is $\sqrt{3}$. Thus $\sin30°=1/2$, $\cos30°=\sqrt{3}/2$, $\tan30°=1/\sqrt{3}=\sqrt{3}/3$. Cofunction symmetry gives the 60° values.

4.3 Core Exact-Value Table

θ	Radians	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	π/6	1/2	√3/2	√3/3
45°	π/4	√2/2	√2/2	1
60°	π/3	√3/2	1/2	√3
90°	π/2	1	0	undefined

4.4 The Square-Root Pattern

For $\theta=0°,30°,45°,60°,90°$, the sine numerators follow $\sqrt{0},\sqrt{1},\sqrt{2},\sqrt{3},\sqrt{4}$ over 2. Cosine uses the same list in reverse. This is a memory device backed by the triangle derivations.

PART I • FOUNDATIONS

5. Reference Angles and All Four Quadrants

Find magnitude first, then attach the correct sign

5.1 Four-Step Procedure

- Reduce the angle to one revolution if needed.
- Locate the quadrant.
- Find the acute reference angle.
- Use the exact reference value and apply the quadrant sign.

5.2 Worked Examples

Expression	Reference Angle	Sign	Exact Result
sin150°	30°	positive in II	1/2
cos150°	30°	negative in II	−√3/2
tan225°	45°	positive in III	1
sin330°	30°	negative in IV	−1/2
cos(−60°)	60°	cosine is even	1/2

5.3 Even and Odd Symmetry

$$\cos(-\theta)=\cos\theta,\ \sin(-\theta)=-\sin\theta,\ \tan(-\theta)=-\tan\theta$$

5.4 Undefined Tangent

Because $\tan\theta=\sin\theta/\cos\theta$, tangent is undefined where $\cos\theta=0$: $\theta=\pi/2+\pi k$. A hand answer should say undefined, not infinity.

PART II • EXACT AND APPROXIMATE METHODS

6. The Identity Toolkit

Transform difficult angles into known information

6.1 Fundamental Identities

Family	Identity
Pythagorean	$\sin^2\theta+\cos^2\theta=1$
Quotient	$\tan\theta=\sin\theta/\cos\theta$
Reciprocal	$\csc\theta=1/\sin\theta$; $\sec\theta=1/\cos\theta$; $\cot\theta=1/\tan\theta$
Cofunction	$\sin(\pi/2-\theta)=\cos\theta$; $\cos(\pi/2-\theta)=\sin\theta$
Even / odd	$\cos(-\theta)=\cos\theta$; $\sin(-\theta)=-\sin\theta$; $\tan(-\theta)=-\tan\theta$

6.2 Recovering Missing Ratios

If $\sin\theta=3/5$ and θ is acute, $\cos\theta=\sqrt{(1-9/25)}=4/5$ and $\tan\theta=(3/5)/(4/5)=3/4$. If the quadrant is specified, select the sign before finalizing the root.

6.3 Identity Checks

A proposed pair (s,c) can represent sine and cosine of one angle only if $s^2+c^2=1$ and the signs match a common quadrant. This provides a fast no-calculator consistency check.

PART II • EXACT AND APPROXIMATE METHODS

7. Sum, Difference, Double, and Half Angles

Manufacture new exact values from familiar ones

Name	Formula
Sine sum	$\sin(A+B)=\sin A\cos B+\cos A\sin B$
Sine difference	$\sin(A-B)=\sin A\cos B-\cos A\sin B$
Cosine sum	$\cos(A+B)=\cos A\cos B-\sin A\sin B$
Cosine difference	$\cos(A-B)=\cos A\cos B+\sin A\sin B$
Tangent sum	$\tan(A+B)=(\tan A+\tan B)/(1-\tan A\tan B)$
Double angle	$\sin2\theta=2\sin\theta\cos\theta$; $\cos2\theta=\cos^2\theta-\sin^2\theta$
Half angle	$\sin(\theta/2)=\pm\sqrt{[(1-\cos\theta)/2]}$; $\cos(\theta/2)=\pm\sqrt{[(1+\cos\theta)/2]}$

7.1 Exact Example: 15°

$$\sin15°=\sin(45°-30°)=(\sqrt{6}-\sqrt{2})/4$$

$$\cos15°=(\sqrt{6}+\sqrt{2})/4,\ \tan15°=2-\sqrt{3}$$

7.2 Exact Example: 75°

Because $75°=45°+30°$, $\sin75°=(\sqrt{6}+\sqrt{2})/4$, $\cos75°=(\sqrt{6}-\sqrt{2})/4$, and $\tan75°=2+\sqrt{3}$.

7.3 Exact Example: 22.5°

Use the half-angle formulas with $\theta=45°$. Since 22.5° is in Quadrant I, choose positive roots:

$$\sin22.5°=\sqrt{(2-\sqrt{2})/2},\ \cos22.5°=\sqrt{(2+\sqrt{2})/2}$$

SIGN WARNING
The ± in a half-angle formula is decided by the quadrant of θ/2, not by personal choice.

PART II • EXACT AND APPROXIMATE METHODS

8. A Hand Strategy for Any Angle

Reduce, decompose, approximate, and check

8.1 Decision Ladder

Question	If Yes	If No
Is θ a core special angle?	Use the exact table.	Find its reference angle.
Can it be formed from 30°,45°,60°,90°?	Use sum, difference, or half-angle identities.	Choose an approximation method.
Is a nearby table value known?	Interpolate with stated error limits.	Use Bhaskara or a power series.
Is tan requested?	Often compute sin and cos first, then divide.	Keep exact forms as long as possible.

8.2 Exact Decomposition Examples

Angle	Decomposition	Result
105°	60°+45°	$\sin 105° = (\sqrt{6}+\sqrt{2})/4$
165°	180°−15°	$\sin 165° = \sin 15° = (\sqrt{6}-\sqrt{2})/4$
225°	180°+45°	$\cos 225° = -\sqrt{2}/2$
330°	360°−30°	$\tan 330° = -\sqrt{3}/3$

8.3 Order-of-Magnitude Checks

Every sine and cosine lies from −1 to 1. Tangent can have any real value but changes rapidly near $90° + 180°k$. In Quadrant I, $\sin\theta$ increases and $\cos\theta$ decreases. These facts detect many copied signs and misplaced decimals.

9. Building Trigonometric Tables by Hand

Recurrence, interpolation, and controlled computation

9.1 Step Recurrence

If $\sin\Delta$ and $\cos\Delta$ are known, addition formulas advance a table one step:

$$\sin(\theta+\Delta) = \sin\theta\cos\Delta + \cos\theta\sin\Delta$$

$$\cos(\theta+\Delta) = \cos\theta\cos\Delta - \sin\theta\sin\Delta$$

Starting from 0°, one may build values at $\Delta, 2\Delta, 3\Delta, \ldots$ using only multiplication and addition. Rounding accumulates, so periodic exact checkpoints such as 30°,45°,60°, and 90° are valuable.

9.2 Linear Interpolation

If a table gives sin20° and sin21°, estimate sin20.4° by moving 0.4 of the difference from the 20° value. This assumes near-linearity over a short interval. The method is practical but should not be confused with an exact identity.

9.3 Symmetry Reduces the Table

A table from 0° to 90° is enough. Reference angles, signs, even-odd rules, and periodicity generate every other angle. Cosine may be read from sine by $\cos\theta = \sin(90° - \theta)$.

9.4 Difference Tables

Historical human computers often recorded first and second differences so that nearby entries could be interpolated efficiently. Smooth trigonometric functions change gradually, making difference columns effective when the step size is small.

10. Bhaskara's Sine Approximation

A memorable rational formula for degrees from 0° to 180°

A classical approximation associated with Bhaskara I estimates sine directly in degrees. For $0° \le \theta \le 180°$:

$$\sin\theta \approx 4\theta(180-\theta) / [40500-\theta(180-\theta)]$$

10.1 Example: sin20°

$$\sin 20° \approx 4\cdot 20\cdot 160 / [40500-20\cdot 160] = 12800/37300 = 128/373$$

Long division gives approximately 0.343. A more refined method gives about 0.342, so the formula is close and easy to compute but not exact.

10.2 Exact Checkpoints

At 0°, 90°, and 180°, the formula gives 0,1,0 exactly. At intermediate angles it is an approximation. Use quadrant and symmetry rules before applying it outside 0° to 180°.

10.3 Strengths and Limits

Strength	Limitation
Uses only integer arithmetic and one division.	Does not preserve exact radical forms.
Easy to memorize and apply in degrees.	Accuracy varies with angle.
Excellent for checking rough magnitude.	Not a substitute for a certified high-precision method.

PART II • EXACT AND APPROXIMATE METHODS

11. Power Series Without a Calculator

Turn radians into polynomial arithmetic

11.1 Core Series

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$$

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots$$

These formulas require radians. Reduce x near zero using symmetry and periodicity first. Alternating terms shrink rapidly for modest |x|, and the first omitted term bounds the error when term magnitudes decrease.

11.2 Example: sin20°

Convert $20° = \pi/9$ and use $\pi \approx 22/7$, so $x \approx 22/63 \approx 0.34921$. Then:

$$\sin x \approx x - x^3/6 + x^5/120 \approx 0.3422$$

The next term $x^7/5040$ is below one millionth, so the series computation is much more accurate than the input approximation $\pi \approx 22/7$ allows. Input error and arithmetic rounding must both be considered.

11.3 Example: cos20° and tan20°

$$\cos x \approx 1 - x^2/2 + x^4/24 \approx 0.9397$$

$$\tan 20° = \sin 20°/\cos 20° \approx 0.3422/0.9397 \approx 0.3641$$

11.4 Small-Angle Estimates

For small radian x, $\sin x \approx x$, $\tan x \approx x$, and $\cos x \approx 1 - x^2/2$. At $1° \approx 0.01745$ radians, $\sin 1° \approx 0.01745$. These are approximations, not identities.

PART III • INVERSES, EQUATIONS, AND TRIANGLES

12. Inverse Trigonometric Functions

Ratios return principal angles

Forward function: angle → ratio **Inverse function: ratio → selected angle**

Restricted ranges make inverse trigonometric functions single-valued.

12.1 Domain and Principal Range

Function	Input Domain	Principal Output Range
arcsin x or $\sin^{-1}x$	$-1 \le x \le 1$	$-\pi/2 \le \theta \le \pi/2$
arccos x or $\cos^{-1}x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
arctan x or $\tan^{-1}x$	all real x	$-\pi/2 < \theta < \pi/2$

12.2 Inverse Is Not Reciprocal

Inverse Function	Reciprocal Function
$\sin^{-1}x$=arcsin x: returns an angle	$\csc x = 1/\sin x$: returns a ratio
$\cos^{-1}x$=arccos x	$\sec x = 1/\cos x$
$\tan^{-1}x$=arctan x	$\cot x = 1/\tan x$

12.3 Exact Inverse Values

Expression	Question Asked	Principal Answer
arcsin(1/2)	Which principal angle has sine 1/2?	π/6 or 30°
arccos(−√2/2)	Which angle from 0 to π has this cosine?	3π/4 or 135°
arctan(−√3)	Which angle from −π/2 to π/2 has this tangent?	−π/3 or −60°

PART III • INVERSES, EQUATIONS, AND TRIANGLES

13. Finding Inverse Values by Hand

Exact recognition, triangles, series, and refinement

13.1 Exact Recognition

First compare the ratio with the exact table. Use signs and the inverse function's principal range. For example, arcsin(−√3/2)=−π/3, while arccos(−√3/2)=5π/6.

13.2 Triangle Construction

To interpret arctan(3/4), draw a right triangle with opposite 3 and adjacent 4. The hypotenuse is 5, so the same angle has sine 3/5 and cosine 4/5. The angle is exact as arctan(3/4), even if its degree measure is approximated.

13.3 Arctangent Series

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \ldots \text{ for } |x| \le 1$$

For x=1/2: 1/2−1/24+1/160−1/896≈0.4635 radians. Multiplying by 180/π with π≈22/7 gives about 26.55°, close to the familiar 3-4-5 triangle angle.

13.4 Arcsine Series

$$\arcsin x = x + x^3/6 + 3x^5/40 + 5x^7/112 + \dots$$

The series works best when $|x|$ is not close to 1. For larger magnitudes, use identities or solve through arctangent with a constructed triangle. For example, if $y=\sin\theta$ and θ is in the principal range, $\tan\theta = y/\sqrt{(1-y^2)}$.

13.5 Newton Refinement

To solve $\sin\theta = y$ from a starting guess θ_0, use:

$$\theta_{n+1} = \theta_n - (\sin\theta_n - y)/\cos\theta_n$$

Each sine and cosine evaluation can come from the series in Chapter 11. The method is powerful but requires a good starting interval and care near $\cos\theta = 0$.

PART III • INVERSES, EQUATIONS, AND TRIANGLES

14. Solving Trigonometric Equations

Principal inverse values are the beginning, not the whole solution

14.1 General Solution Forms

Equation	All Real Solutions
$\sin\theta = a$	$\theta = \arcsin a + 2\pi k$ or $\theta = \pi - \arcsin a + 2\pi k$
$\cos\theta = a$	$\theta = \pm\arccos a + 2\pi k$
$\tan\theta = a$	$\theta = \arctan a + \pi k$

14.2 Example: $\sin\theta = 1/2$

The principal arcsine is $\pi/6$. Sine is also positive in Quadrant II, giving $5\pi/6$. Thus $\theta = \pi/6 + 2\pi k$ or $5\pi/6 + 2\pi k$.

14.3 Example: $\cos\theta = -\sqrt{2}/2$

The reference angle is $\pi/4$ and cosine is negative in Quadrants II and III. On $0 \le \theta < 2\pi$, the solutions are $3\pi/4$ and $5\pi/4$. The general solutions are $\theta = \pm 3\pi/4 + 2\pi k$, equivalently those two families.

14.4 Factoring Trigonometric Expressions

Solve $2\sin^2\theta - \sin\theta = 0$ by factoring: $\sin\theta(2\sin\theta - 1) = 0$. Therefore $\sin\theta = 0$ or $\sin\theta = 1/2$. On $0 \le \theta < 2\pi$, $\theta = 0, \pi, \pi/6, 5\pi/6$.

CHECK THE INTERVAL
Inverse notation returns a principal angle. An equation may require additional angles, coterminal families, or restrictions to a stated interval.

PART III • INVERSES, EQUATIONS, AND TRIANGLES

15. Solving Triangles by Hand

Use ratios for right triangles and laws for oblique triangles

15.1 Right Triangle

If opposite=7 and adjacent=24, the hypotenuse is 25. Then $\theta = \arctan(7/24)$. The arctangent series gives an approximation, while the exact angle remains $\arctan(7/24)$. The other acute angle is $90° - \theta$.

15.2 Law of Sines

$$a/\sin A = b/\sin B = c/\sin C$$

Use when an angle-side opposite pair is known. The ambiguous SSA case may produce zero, one, or two triangles because $\sin B = \sin(180° - B)$.

15.3 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Use for SAS or SSS information. When C=90°, cosC=0 and the formula becomes the Pythagorean theorem.

15.4 Area

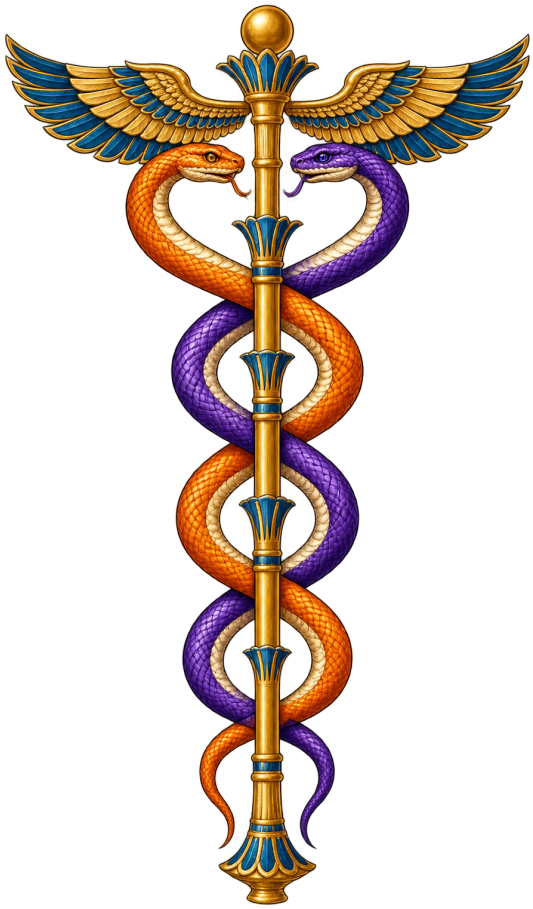
Area = ½ab sinC

If C is special, the area can remain exact. For a=8, b=10, C=30°, Area=½·8·10·½=20 square units.

PART IV • SYMBOL, HELIX, AND HISTORY

16. The Caduceus, Tehuti, and the Two Wave Paths

A modern mathematical analogy using periodic motion around a central axis



Visual reading

The golden staff acts as a central axis. The orange and purple serpents repeatedly cross, separate, reach maximum distance, return, and cross again. Their side-to-side positions form periodic traces.

Mathematical reading

A helix around a staff has perpendicular coordinates Rcos t and Rsin t. When projected onto flat planes, these coordinates become cosine and sine waves.

Cultural caution

This page uses a modern educational synthesis. It does not claim that ancient artists encoded today's sine and cosine notation in the symbol.

16.1 Historical Naming

The classical caduceus is most directly associated with the Greek Hermes and Roman Mercury. Tehuti, also written Djehuty and known in Greek as Thoth, was associated with writing, reckoning, knowledge, and the moon. In Hellenistic Egypt, Thoth and Hermes were identified through religious and philosophical syncretism, contributing to the later figure Hermes Trismegistus. Calling the illustration a 'Staff of Tehuti' is therefore best treated as a symbolic Hermetic-Egyptian interpretation, not as the standard archaeological name of one original object.

16.2 What the Image Shows Well

Visual Feature	Mathematical Analogy
Straight staff	Central axis or equilibrium line
Repeated coils	Periodicity
Maximum left/right reach	Amplitude
Crossings	Zeros or equal projected positions
Opposing serpent positions	Phase separation
Upward progression	Independent variable or parameter t

PART IV • SYMBOL, HELIX, AND HISTORY

17. Helices, Projections, and Phase

The precise geometry behind the serpent-wave analogy

17.1 One Helix Contains Both Sine and Cosine

A circular helix of radius R rising at rate c can be parameterized by:

$$\mathbf{r(t) = (R \cos t, R \sin t, ct)}$$

Looking from one side reveals x(t)=Rcos t. Looking from a perpendicular side reveals y(t)=Rsin t. The two functions are not separate motions; they are perpendicular coordinate shadows of the same rotation.

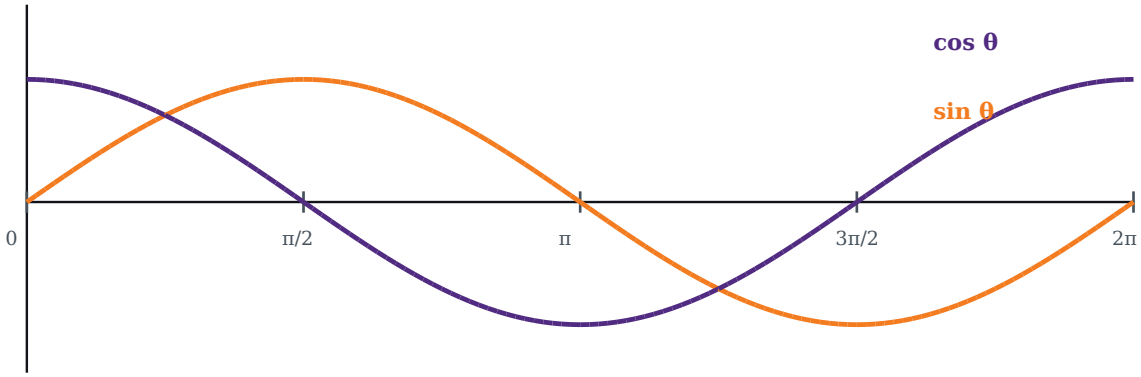
17.2 A Second Opposing Serpent

$$\mathbf{r_2(t) = (R \cos(t+\pi), R \sin(t+\pi), ct)}$$

A second serpent on the opposite side is shifted by π radians: its projection is the negative of the first. A stylized artist may choose a different phase shift. Sine and cosine themselves differ by π/2, since cos t=sin(t+π/2).

17.3 A Flat Wave Is a Projection

SINE AND COSINE: SAME SHAPE, QUARTER-CYCLE SHIFT



Unwrapping or projecting a spiral does not preserve every visual detail, but it preserves periodic coordinate motion. The analogy is mathematically useful precisely when the axis, amplitude, period, and phase are defined.

17.4 Limits of the Analogy

Real serpent artwork may vary coil spacing, body thickness, crossing order, and phase. Such an image is not a measured graph. The trigonometric model is an idealization chosen to explain repetition and projection.

PART IV • SYMBOL, HELIX, AND HISTORY

18. Article: Trigonometry Without Modern Calculators

How geometry, tables, identities, interpolation, and human computers solved angle problems

18.1 Chords Before Modern Sines

Ancient Greek astronomy often worked with chords of circles rather than the modern sine function. Ptolemy's Almagest presented a detailed chord table and geometric theorems for combining arcs. A chord of central angle 2θ in a circle of radius R has length 2Rsinθ, so chord methods and sine methods are closely related.

18.2 Indian Sine Tables

Indian mathematicians developed half-chord functions corresponding to sine and created tables through finite-difference and recursive procedures. Aryabhata's work is a major landmark. Later writers refined interpolation and approximation, including the rational sine formula associated with Bhaskara I.

18.3 Islamic Astronomy and High-Precision Tables

Mathematicians working in Arabic and Persian scholarly traditions expanded plane and spherical trigonometry, developed tangent and cotangent tables, improved interpolation, and applied the results to astronomy, geography, timekeeping, and direction finding. Al-Kashi is famous for extremely accurate numerical work.

18.4 European Tables, Prosthaphaeresis, and Logarithms

Before electronic calculators, printed sine, cosine, tangent, and logarithm tables were routine scientific instruments. Prosthaphaeresis identities converted products into sums of trigonometric values. Logarithms later converted multiplication into addition, and slide rules embodied logarithmic scaling mechanically.

18.5 The Human-Computer Workflow

A reliable computation was divided into stages: reduce the angle, look up or derive a nearby value, interpolate, carry guard digits, determine the quadrant sign, and perform an independent check. Tables required skilled construction, copying, correction, and maintenance. Accuracy was a social process as well as an arithmetic one.

18.6 Why These Methods Still Matter

A calculator can return $\sin 237°$ immediately, but it may not reveal that the answer must be negative, that the reference angle is $57°$, or that the magnitude should be near $\sin 60°$. Hand methods train structure, error awareness, exactness, and the ability to recognize an impossible machine output.

Pre-Electronic Tool	Purpose	Modern Lesson
Special triangles	Produce exact ratios.	Derive before memorizing.
Chord / sine tables	Store expensive calculations.	Use verified reference data.
Identities	Transform unknown angles into known ones.	Algebra can replace direct measurement.
Finite differences	Interpolate between table entries.	Track approximation error.
Logarithms / slide rules	Simplify multiplication and powers.	Estimate scale and significant digits.
Independent recomputation	Catch transcription and arithmetic errors.	Verification is a separate task.

PART V • APPLICATIONS AND ASSESSMENT

19. Applications

Surveying, waves, rotation, navigation, and periodic systems

19.1 Height by Shadow

If a vertical object casts a shadow s and the Sun's elevation is θ, then height h=s tanθ. When θ=45°, h=s exactly. For an unfamiliar θ, approximate tanθ from sinθ/cosθ and state the measurement uncertainty.

19.2 Components of a Vector

$$F_x=F\cos\theta,\ F_y=F\sin\theta$$

A force, velocity, or displacement of magnitude F decomposes into perpendicular components. The Pythagorean identity guarantees $F_x^2+F_y^2=F^2$.

19.3 Waves and Phase

$$y(t)=A\sin(\omega t+\varphi)$$

A is amplitude, ω controls period T=2π/ω, and φ is phase. The caduceus analogy visualizes repeated phase relationships but should be replaced by measured data for scientific modeling.

19.4 Circular and Rotational Motion

$$x(t)=R\cos\omega t,\ y(t)=R\sin\omega t$$

These equations describe uniform circular motion. Adding z(t)=ct produces a helix, directly connecting unit-circle coordinates to the serpent-around-staff model.

19.5 Navigation and Surveying

Bearings, triangulation, and celestial observations use angular ratios to infer inaccessible distances. Before electronic instruments, tables and interpolation converted observed angles into working numbers.

PART V • APPLICATIONS AND ASSESSMENT

20. Classroom Investigations

Construct, graph, compute, compare, and critique

Investigation A: Rebuild the Exact Table

Draw the 45°-45°-90° and 30°-60°-90° triangles, derive every ratio, rationalize denominators, and assemble the five-angle table without copying it.

Investigation B: Unit-Circle Paper Model

Mark angles every 15° on a circle. Project each point to the axes. Plot the vertical projections as sine and the horizontal projections as cosine against angle.

Investigation C: Compare Hand Approximations

Estimate sin20° with Bhaskara's formula and with three nonzero Taylor terms. Compare the results, arithmetic workload, and available error statement.

Investigation D: Two-Serpent Phase Model

Wrap two colored strings around a cylinder. Place them opposite each other, then repeat with a quarter-turn shift. Photograph or sketch the side projections. Identify amplitude, period, zeros, and phase.

Investigation E: Inverse Function Cards

Create cards pairing an exact ratio with arcsin, arccos, and arctan outputs. Include principal ranges so the same ratio can produce different inverse answers.

Reusable Hand-Computation Sheet

Angle / Ratio	Method	Exact Form	Approximation	Sign / Range Check
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____

PART V • APPLICATIONS AND ASSESSMENT

21. Problem Bank

Foundations, exact values, approximations, inverse functions, and applications

A. Angles and Foundations

- 1. Convert 225° to radians.
- 2. Convert $11\pi/6$ to degrees.
- 3. Find a positive and a negative coterminal angle for 40°.
- 4. Find the reference angle and quadrant of 230°.
- 5. State the signs of sin, cos, and tan in Quadrant IV.

B. Exact Values

- 6. Find sin60°, cos60°, and tan60°.
- 7. Find sin135°, cos135°, and tan135°.
- 8. Find sin315°, cos315°, and tan315°.
- 9. Find sin15° and cos15° exactly.
- 10. Find tan75° exactly.
- 11. Find sin22.5° exactly.

C. Identities and Ratios

- 12. If $\sin\theta=5/13$ and θ is acute, find $\cos\theta$ and $\tan\theta$.
- 13. If $\cos\theta=-12/13$ and θ is in Quadrant II, find $\sin\theta$ and $\tan\theta$.
- 14. Verify $\sin^2 30°+\cos^2 30°=1$.
- 15. Simplify $(1-\cos^2\theta)/\sin\theta$ where defined.

D. Hand Approximations

- 16. Use Bhaskara's formula to estimate sin30° and compare with the exact value.
- 17. Use $x-x^3/6+x^5/120$ to estimate $\sin(\pi/6)$, using $\pi\approx 22/7$.
- 18. Use $1-x^2/2+x^4/24$ to estimate cos0.2 radians.
- 19. Explain why tan89° should be large and positive without calculating it.

E. Inverse Functions and Equations

- 20. Evaluate $\arcsin(-1/2)$.
- 21. Evaluate $\arccos(-1/2)$.
- 22. Evaluate $\arctan(1)$.
- 23. Use four terms of the arctangent series to estimate $\arctan(1/2)$.
- 24. Solve $\sin\theta=-\sqrt{2}/2$ on $0\le\theta<2\pi$.
- 25. Solve $\cos\theta=1/2$ for all real θ.
- 26. Solve $2\cos^2\theta-1=0$ on $0\le\theta<2\pi$.

F. Applications and Reasoning

- 27. A 20 m rope makes a 30° angle above level ground. Find its horizontal and vertical components exactly.
- 28. Find the area of a triangle with sides 6 and 10 enclosing 45°.
- 29. Explain precisely how a helix produces sine and cosine projections.
- 30. Explain why two opposite serpents are π out of phase, while sine and cosine are π/2 out of phase.
- 31. Describe two safeguards used by human computers before electronic calculators.
- 32. Write one sentence distinguishing historical evidence from the modern caduceus analogy.

PART V • APPLICATIONS AND ASSESSMENT

22. Answer Key and Solution Notes

Essential reasoning is shown; expand proofs and written arithmetic as assigned

Quick Diagnostic

1. 5π/6. 2. 315°. 3. sin=opp/hyp, cos=adj/hyp, tan=opp/adj. 4. (cosθ,sinθ). 5. 1. 6. 1/2. 7. 1. 8. Negative. 9. π/6 or 30°. 10. $\sin^{-1}$ is arcsin; csc is the reciprocal 1/sin.

Solutions 1-11

No.	Solution
1	225°·π/180=5π/4.
2	11π/6·180/π=330°.
3	400° and −320°, among infinitely many.
4	Quadrant III; reference angle 50°.
5	sin negative, cos positive, tan negative.
6	√3/2, 1/2, √3.
7	√2/2, −√2/2, −1.
8	−√2/2, √2/2, −1.
9	sin15°=(√6−√2)/4; cos15°=(√6+√2)/4.
10	tan75°=2+√3.
11	sin22.5°=√(2−√2)/2.

Solutions 12-23

No.	Solution
12	A 5-12-13 triangle gives cosθ=12/13 and tanθ=5/12.
13	Quadrant II makes sine positive: sinθ=5/13 and tanθ=−5/12.
14	$(1/2)^2+(\sqrt{3}/2)^2=1/4+3/4=1$.
15	$(1-\cos^2\theta)/\sin\theta=\sin^2\theta/\sin\theta=\sin\theta$ where sinθ≠0.
16	Bhaskara: 4·30·150/[40500−4500]=18000/36000=1/2, exact here.
17	With x≈11/21, $x-x^3/6+x^5/120\approx0.5002$; accuracy is limited by π≈22/7 and truncation.
18	$1-0.2^2/2+0.2^4/24=1-0.02+0.0000667\approx0.98007$.
19	cos89° is small positive while sin89° is near 1, so tan=sin/cos is large positive.
20	−π/6.
21	2π/3.
22	π/4.
23	1/2−1/24+1/160−1/896≈0.4635 radians.

Solutions 24-32

No.	Solution
24	Reference angle π/4; sine negative in III and IV: θ=5π/4,7π/4.
25	θ=±π/3+2πk.
26	$\cos^2\theta=1/2$, so $\cos\theta=\pm\sqrt{2}/2$: θ=π/4,3π/4,5π/4,7π/4.
27	Horizontal=20cos30°=10√3 m; vertical=20sin30°=10 m.
28	Area=½·6·10·sin45°=15√2 square units.
29	For r(t)=(Rcost,Rsint,ct), perpendicular side projections are Rcost and Rsint.
30	Opposite positions add phase π and negate both coordinates. Cosine equals sine shifted only π/2.
31	Examples: reference-angle sign checks, difference tables, guard digits, independent recomputation, exact checkpoints.
32	The symbol is historical, but interpreting its serpent paths as plotted sine and cosine is a modern teaching model.

PART V • APPLICATIONS AND ASSESSMENT

23. Formula and Exact-Value Reference

A compact desk sheet for computation without a modern calculator

Topic	Formula
Triangle ratios	sin=opp/hyp; cos=adj/hyp; tan=opp/adj
Unit circle	(x,y)=(cosθ,sinθ); tanθ=y/x
Pythagorean	$\sin^2\theta+\cos^2\theta=1$
Degrees / radians	radians=degrees·π/180; degrees=radians·180/π
Sum identities	sin(A+B)=sinAcosB+cosAsinB; cos(A+B)=cosAcosB−sinAsinB
Double angle	$\sin2\theta=2\sin\theta\cos\theta$; $\cos2\theta=\cos^2\theta-\sin^2\theta$
Bhaskara sine	$\sin\theta\approx4\theta(180-\theta)/[40500-\theta(180-\theta)]$ for degrees 0 to 180
Series	$\sin x=x-x^3/6+x^5/120-\ldots$; $\cos x=1-x^2/2+x^4/24-\ldots$
Inverse tangent series	$\arctan x=x-x^3/3+x^5/5-x^7/7+\ldots$
Helix	r(t)=(Rcost,Rsint,ct)

Core Exact Values

θ	0°	30°	45°	60°	90°
sinθ	0	1/2	√2/2	√3/2	1
cosθ	1	√3/2	√2/2	1/2	0
tanθ	0	√3/3	1	√3	undefined

Inverse Principal Ranges

Inverse	Domain	Range
arcsin	[−1,1]	[−π/2,π/2]
arccos	[−1,1]	[0,π]
arctan	all real	(−π/2,π/2)

PART V • APPLICATIONS AND ASSESSMENT

24. Glossary

Vocabulary for angles, functions, waves, approximation, and inverses

Amplitude Maximum displacement from a sinusoid's centerline.	Arccos Inverse cosine with principal range [0,π].

Arcsin Inverse sine with principal range $[-\pi/2,\pi/2]$.	Arctan Inverse tangent with principal range $(-\pi/2,\pi/2)$.
Cofunction A complementary-angle partner, such as sine and cosine.	Coterminal angles Angles sharing a terminal side.
Helix A three-dimensional curve combining circular motion with steady axial motion.	Identity An equation true for every value in its domain.
Interpolation Estimating between known table entries.	Inverse function A function that reverses an input-output relationship on a restricted domain.
Period Smallest positive input shift that repeats a function.	Phase Horizontal position of a periodic cycle relative to a reference.
Principal value The single selected output of a multivalued inverse relationship.	Radian Angle subtending an arc equal in length to the radius.
Reference angle Positive acute angle between a terminal side and the x-axis.	Sinusoid A sine-shaped periodic curve, including shifted and scaled forms.
Tangent The ratio $\sin\theta/\cos\theta$, where $\cos\theta \neq 0$.	Unit circle Circle of radius 1 centered at the origin.

PART V • APPLICATIONS AND ASSESSMENT

25. References and Further Study

Historical, cultural, mathematical, and instructional sources

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2. Aryabhata. Aryabhatiya. Foundational Indian astronomical mathematics and sine-table methods.

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11. Fowden, Garth. The Egyptian Hermes. Princeton University Press. Hellenistic and late-antique Thoth-Hermes syncretism.

12. OpenStax. Precalculus 2e. Open educational reference for trigonometric functions, identities, equations, and inverses.

13. NIST. Digital Library of Mathematical Functions. Authoritative reference for trigonometric functions and series.

Interpretive Note

The caduceus illustration is original artwork created for this textbook. Its trigonometric interpretation is a contemporary educational analogy. The historical discussion distinguishes the classical emblem of Hermes from the later Thoth-Hermes identification and does not present the image as archaeological proof of ancient sine-wave encoding.

Suggested Next Topics

- Spherical trigonometry
- Fourier series and harmonics
- Complex exponentials and Euler's formula
- CORDIC algorithms

- Navigation and geodesy
- Phase analysis in electricity and sound

PART V • APPLICATIONS AND ASSESSMENT

26. Index, Mastery Checklist, and Reflection

A final map of concepts and a record of learning

Index

Entry	Location	Entry	Location
angle conversion	Ch. 1	interpolation	Ch. 9,18
arccos	Ch. 12-13	inverse functions	Ch. 12-14
arcsin	Ch. 12-13	Pythagorean identity	Ch. 6
arctan	Ch. 12-13	quadrant signs	Ch. 3,5
Bhaskara formula	Ch. 10	reference angle	Ch. 1,5
caduceus	Ch. 16-17	right triangle	Ch. 2,15
cosine	Ch. 2-11	series	Ch. 11,13
degrees / radians	Ch. 1	special angles	Ch. 4
helix	Ch. 17,19	tangent	Ch. 2-11
historical methods	Ch. 18	Tehuti / Thoth	Ch. 16
identities	Ch. 6-8	unit circle	Ch. 3

Student Mastery Checklist

I Can...	Developing	Secure	Teach It
Convert degrees and radians.	☐	☐	☐
Derive and use special-angle values.	☐	☐	☐
Use reference angles and quadrant signs.	☐	☐	☐
Apply identities to find exact values.	☐	☐	☐
Approximate sine and cosine by hand.	☐	☐	☐
Distinguish inverse and reciprocal functions.	☐	☐	☐
Find exact and approximate inverse values.	☐	☐	☐
Solve trigonometric equations and triangles.	☐	☐	☐
Explain the helix and caduceus analogy accurately.	☐	☐	☐

Reflection

Prompt	Response
Which no-calculator method is most useful to you?	_____ _____
What exact form should never be rounded too early?	_____ _____
How does projection turn a helix into sine and cosine?	_____ _____
How will you distinguish symbolism from historical evidence?	_____ _____

FINAL PRINCIPLE
A strong hand solution identifies the angle system, preserves exact structure, chooses an appropriate method, applies the correct sign or principal range, and verifies the result independently.

END OF TEXTBOOK
www.FreeMathTutor.com